

**THE INTEGRALS IN GRADSHTEYN AND RYZHIK.  
PART 8: COMBINATIONS OF POWERS, EXPONENTIALS AND  
LOGARITHMS.**

VICTOR H. MOLL, JASON ROSENBERG, ARMIN STRAUB, AND PAT WHITWORTH

ABSTRACT. We describe some examples of integrals from the table of Gradshteyn and Ryzhik where the integrand is a combination of powers, exponentials and logarithms. The expressions for some of these integrals involve the Stirling numbers of the first kind.

### 1. INTRODUCTION

The uninitiated reader of the table of integrals by I. S. Gradshteyn and I. M. Ryzhik [4] will surely be puzzled by choice of integrands. In this note we provide an elementary proof of the evaluation **4.353.3**

$$(1) \quad \int_0^1 (ax + n + 1)x^n e^{ax} \ln x \, dx = e^a \sum_{k=0}^n (-1)^{k-1} \frac{n!}{(n-k)!a^{k+1}} + (-1)^n \frac{n!}{a^{n+1}}.$$

We also consider the integrals

$$(2) \quad q_n := \int_0^1 x^n e^{-x} \ln x \, dx$$

and the companion family

$$(3) \quad p_n := \int_0^1 x^n e^{-x} \, dx.$$

The integral  $q_n$  corresponds to the case  $a = -1$  in (1). Section 3 provides closed-form expressions for  $p_n$  and  $q_n$ . Section 4 considers the generalization

$$(4) \quad P_n(a) = \int_0^1 x^n e^{-ax} \, dx \text{ and } Q_n(a) = \int_0^1 x^n e^{-ax} \ln x \, dx.$$

The main result of this section is the closed-form expressions

$$(5) \quad P_n(a) := \int_0^1 x^n e^{-ax} \, dx = \frac{n!}{a^{n+1}} \left( 1 - e^{-a} \sum_{k=0}^n \frac{a^k}{k!} \right),$$

and

$$Q_n(a) := \int_0^1 x^n e^{-ax} \ln x \, dx = \frac{n!}{a^{n+1}} \left[ \sum_{k=1}^n \frac{1}{k} \left( 1 - e^{-a} \sum_{j=0}^{k-1} \frac{a^j}{j!} \right) + a Q_0(a) \right],$$

---

2000 *Mathematics Subject Classification.* Primary 33.

*Key words and phrases.* Powers and exp-log.

The authors wish to the partial support of nsf-ccli 0633223.

where

$$(6) \quad Q_0(a) = \int_0^1 e^{-ax} \ln x \, dx = -\frac{1}{a} (\gamma + \ln a + \Gamma(0, a)),$$

and  $\Gamma(0, a)$  is the incomplete gamma function defined by

$$(7) \quad \Gamma(a, x) := \int_x^\infty t^{a-1} e^{-t} \, dt.$$

## 2. THE EVALUATION OF 4.353.3

The identity

$$(1) \quad \frac{d}{dx} (x^{n+1} e^{ax}) = (ax + n + 1) x^n e^{ax}$$

and integration by parts yield

$$(2) \quad \int_0^1 (ax + n + 1) x^n e^{ax} \ln x \, dx = - \int_0^1 x^n e^{ax} \, dx.$$

This last integral appears as **3.351.1** in [4]. We have obtained a closed-form expression for it in [2]. A new proof is presented in Section 4.

A closed form expression for the right hand side of (2) is obtained from

$$(3) \quad \int_0^1 x^n e^{ax} \, dx = \left( \frac{d}{da} \right)^n \frac{e^a - 1}{a}.$$

The symbolic evaluation of (3) for small values of  $n \in \mathbb{N}$  suggests the existence of a polynomial  $P_n(a)$  such that

$$(4) \quad \int_0^1 x^n e^{ax} \, dx = \frac{(-1)^{n+1} n!}{a^{n+1}} + \frac{P_n(a)}{a^{n+1}} e^a.$$

The next lemma confirms the existence of this polynomial.

**Lemma 2.1.** *The function  $P_n(a)$  defined by*

$$(5) \quad P_n(a) = a^{n+1} e^{-a} \left( \left( \frac{d}{da} \right)^n \frac{e^a - 1}{a} - \frac{(-1)^{n+1} n!}{a^{n+1}} \right)$$

*is a polynomial of degree  $n$ .*

*Proof.* Let  $D = \frac{d}{da}$ . Then  $D^{n+1} = D(D^n)$  produces the recurrence

$$(6) \quad P_{n+1}(a) = a P_n'(a) + (a - n - 1) P_n(a).$$

The initial condition  $P_0(a) = 1$  and (6) show that  $P_n$  is a polynomial of degree  $n$ .  $\square$

**Theorem 2.2.** *The polynomial*

$$(7) \quad Q_n(a) := (-1)^n P_n(-a)$$

*has positive integer coefficients, written as*

$$(8) \quad Q_n(a) = \sum_{k=0}^n b_{n,k} a^k.$$

These coefficients satisfy

$$(9) \quad \begin{aligned} b_{n+1,0} &= (n+1)b_{n,0} \\ b_{n+1,k} &= (n+1-k)b_{n,k} + b_{n,k-1}, \quad 1 \leq k \leq n \\ b_{n+1,n+1} &= b_{n,n}. \end{aligned}$$

Moreover, the polynomial  $Q_n(a)$  is given by

$$(10) \quad Q_n(a) = n! \sum_{k=0}^n \frac{a^k}{k!}$$

*Proof.* The recurrence (6) yields

$$(11) \quad Q_{n+1}(a) = -aQ'_n(a) + (a+n+1)Q_n(a).$$

The recursion for the coefficients  $b_{n,k}$  follows directly from here. Moreover, it is clear that  $b_{n,n} = 1$  and  $b_{n,0} = n!$ . A little experimentation suggests that  $b_{n,k} = n!/k!$ , and this can be established from (9).  $\square$

This proposition amounts to the evaluation of **3.351.1** in [4]:

$$(12) \quad \int_0^u x^n e^{ax} dx = \frac{(-1)^{n+1} n!}{a^{n+1}} + \frac{e^{au}}{a^{n+1}} \sum_{k=0}^n \frac{n!}{k!} (-1)^{n-k} u^k a^k.$$

The reader will find a proof of this formula in [2].

### 3. A NEW FAMILY OF INTEGRALS

In this section we consider the family of integrals

$$(1) \quad q_n := \int_0^1 x^n e^{-x} \ln x dx,$$

and its companion

$$(2) \quad p_n := \int_0^1 x^n e^{-x} dx.$$

**Lemma 3.1.** *The integrals  $p_n, q_n$  satisfy the recursion*

$$(3) \quad p_{n+1} = (n+1)p_n - e^{-1}$$

$$(4) \quad q_{n+1} = (n+1)q_n + p_n$$

*Proof.* Integrate by parts.  $\square$

The initial conditions are

$$(5) \quad p_0 = 1 - e^{-1} \text{ and } q_0 = \int_0^1 e^{-x} \ln x dx = \gamma - \text{Ei}(-1).$$

Here  $\gamma$  is Euler's constant defined by

$$(6) \quad \gamma := \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{k} - \ln n$$

with integral representation

$$(7) \quad \gamma = \int_0^\infty e^{-x} \ln x dx$$

given as **4.331.1**. The reader will find in [3] a proof of this identity. The second term in (5) is converted into

$$(8) \quad \int_1^\infty e^{-x} \ln x \, dx = \int_1^\infty \frac{e^{-x}}{x} \, dx$$

and this last form is identified as  $\text{Ei}(-1)$ , where  $\text{Ei}$  is the exponential integral defined by

$$(9) \quad \text{Ei}(z) = - \int_{-z}^\infty \frac{e^{-x}}{x} \, dx.$$

In the current context, the value of  $\text{Ei}(-1)$  will be simply one of the terms in the initial condition  $q_0$ .

We determine first an explicit expression for  $p_n$ . The recursion (3) shows the existence of integers  $a_n, b_n$  such that

$$(10) \quad p_n = a_n + b_n e^{-1},$$

with  $a_0 = 1, b_0 = -1$ . From (3) we obtain

$$(11) \quad a_{n+1} + b_{n+1} e^{-1} = (n+1)a_n + [(n+1)b_n - 1] e^{-1}.$$

The irrationality of  $e$  produce the system

$$(12) \quad a_{n+1} = (n+1)a_n, \text{ with } a_0 = 1,$$

$$(13) \quad b_{n+1} = (n+1)b_n - 1, \text{ with } b_0 = -1.$$

The expression  $a_n = n!$  follows directly from (12). To solve (13), define  $B_n := b_n/n!$  and observe that

$$(14) \quad B_{n+1} = B_n - \frac{1}{(n+1)!},$$

that telescopes to

$$(15) \quad b_n = -n! \sum_{k=0}^n \frac{1}{k!}.$$

We have shown:

**Proposition 3.2.** *The integral  $p_n$  in (2) is given by*

$$(16) \quad p_n = \int_0^1 x^n e^{-x} \, dx = \frac{n!}{e} \left( e - \sum_{k=0}^n \frac{1}{k!} \right).$$

We now determine a similar closed-form for  $q_n$ . The recursion (4) shows the existence of integers  $c_n, d_n, f_n$  such that

$$(17) \quad q_n = c_n + d_n e^{-1} + f_n q_0.$$

In order to produce a system similar to (12,13) we will assume that the constants  $1, e^{-1}$  and  $q_0 = -(\gamma + \text{Ei}(-1))$  are linearly independent over  $\mathbb{Q}$ . Under this assumption (4) produces

$$(18) \quad c_{n+1} = (n+1)c_n + n!,$$

$$(19) \quad d_{n+1} = (n+1)c_n - n! \sum_{k=0}^n \frac{1}{k!},$$

$$(20) \quad f_{n+1} = (n+1)f_n,$$

with the initial conditions  $c_0 = 0$ ,  $d_0 = 0$  and  $f_0 = 1$ .

The expression  $f_n = n!$  follows directly from (20). To solve (18) and (19) we employ the following result established in [1].

**Lemma 3.3.** *Let  $a_n$ ,  $b_n$  and  $r_n$  be sequences with  $a_n, b_n \neq 0$ . Assume that  $z_n$  satisfies*

$$(21) \quad a_n z_n = b_n z_{n-1} + r_n, \quad n \geq 1$$

with initial condition  $z_0$ . Then

$$(22) \quad z_n = \frac{b_1 b_2 \cdots b_n}{a_1 a_2 \cdots a_n} \left( z_0 + \sum_{k=1}^n \frac{a_1 a_2 \cdots a_{k-1}}{b_1 b_2 \cdots b_k} r_k \right).$$

We conclude that

$$(23) \quad c_n = n! \sum_{k=1}^n \frac{1}{k},$$

and

$$(24) \quad d_n = -n! \sum_{k=1}^n \frac{1}{k} \sum_{j=0}^{k-1} \frac{1}{j!}.$$

The expression for  $c_n$  shows that they coincide with the Stirling numbers of the first kind:  $c_n = |s(n+1, 2)|$ .

We have established

**Proposition 3.4.** *The integral  $q_n$  in (1) is given by*

$$(25) \quad q_n = \int_0^1 x^n e^{-x} \ln x \, dx = n! \left[ \frac{1}{e} \sum_{k=1}^n \frac{1}{k} \left( e - \sum_{j=0}^{k-1} \frac{1}{j!} \right) + q_0 \right].$$

**Example 3.5.** The expressions for  $p_n$  and  $q_n$  provide the evaluation of **4.351.1** in [4]

$$(26) \quad \int_0^1 (1-x)e^{-x} \ln x \, dx = \frac{1-e}{e},$$

by identifying the integral as  $q_0 - q_1$ . The recurrence (4) shows that

$$(27) \quad q_0 - q_1 = -p_0 = e^{-1} - 1,$$

as claimed.

**Example 3.6.** The evaluation of **4.362.1** in [4]

$$(28) \quad \int_0^1 x e^x \ln(1-x) \, dx = \int_0^1 (1-t) e^{1-t} \ln t \, dt$$

is achieved by observing that this integral is  $e(q_0 - q_1) = 1 - e$ .

## 4. A PARAMETRIC FAMILY

In this section we consider the evaluation of

$$(1) \quad P_n(a) := \int_0^1 x^n e^{-ax} dx$$

$$(2) \quad Q_n(a) := \int_0^1 x^n e^{-ax} \ln x dx.$$

The integrals  $q_n$  considered in Section 3 corresponds to the special case:  $q_n = Q_n(1)$ .

We now establish a recursion for  $Q_n$  by differentiating (2).

**Lemma 4.1.** *The integral  $Q_n(a)$  satisfies the relation*

$$(3) \quad Q_{n+1}(a) = -\frac{d}{da} Q_n(a).$$

To obtain a closed-form expression for  $Q_n(a)$  we need to determine the initial condition

$$(4) \quad Q_0(a) = \int_0^1 e^{-ax} \ln x dx.$$

This is expressed in terms of the *incomplete gamma function* defined in **8.350.1** by

$$(5) \quad \Gamma(a, x) := \int_x^\infty t^{a-1} e^{-t} dt.$$

Observe that  $\Gamma(a, 0) = \Gamma(a)$ , the usual gamma function.

**Lemma 4.2.** *The initial condition  $Q_0(a)$  is given by*

$$(6) \quad Q_0(a) = \int_0^1 e^{-ax} \ln x dx = -\frac{1}{a} (\gamma + \ln a + \Gamma(0, a)).$$

*Proof.* The change of variables  $t = ax$  yields

$$(7) \quad Q_0(a) = \frac{1}{a} \int_0^a e^{-t} \ln t dt - \frac{\ln a}{a} (1 - e^{-a}).$$

Then

$$(8) \quad \int_0^a e^{-t} \ln t dt = \int_0^\infty e^{-t} \ln t dt - \int_a^\infty e^{-t} \ln t dt.$$

The first integral is

$$(9) \quad \int_0^\infty e^{-t} \ln t dt = -\gamma,$$

that simply reflects the fact that  $\gamma = -\Gamma'(1)$ . Integrating by parts yields

$$(10) \quad \int_a^\infty e^{-t} \ln t dt = e^{-a} \ln a + \Gamma(0, a).$$

The formula for  $Q_0(a)$  is established. □

We now determine a closed-form expression for  $P_n(a)$  and  $Q_n(a)$  following the procedure employed in Section 3.

**Lemma 4.3.** *The integrals  $P_n$  and  $Q_n(a)$  satisfy the recursion*

$$(11) \quad P_{n+1}(a) = \frac{1}{a} ((n+1)P_n(a) - e^{-a})$$

$$(12) \quad Q_{n+1}(a) = \frac{1}{a} ((n+1)Q_n(a) + P_n(a)).$$

*The initial conditions are given by*

$$(13) \quad P_0(a) = \frac{1}{a}(1 - e^{-a}), \text{ and } Q_0(a) = -\frac{1}{a}(\gamma + \Gamma(0, a) + \ln a).$$

*Proof.* Integrate by parts. □

We conclude that we can write

$$(14) \quad P_n(a) = A_n(a) - B_n(a)e^{-a},$$

and

$$(15) \quad Q_n(a) = C_n(a) - D_n(a)e^{-a} - E_n(a)(\gamma + \Gamma(0, a) + \ln a).$$

**Lemma 4.4.** *The recursions (11) and (12) imply that*

$$(16) \quad \begin{aligned} A_{n+1}(a) &= \frac{1}{a}(n+1)A_n(a), \\ B_{n+1}(a) &= \frac{1}{a}[(n+1)B_n(a) + 1], \\ C_{n+1}(a) &= \frac{1}{a}[(n+1)C_n(a) + A_n(a)], \\ D_{n+1}(a) &= \frac{1}{a}[(n+1)D_n(a) + B_n(a)], \\ E_{n+1}(a) &= \frac{1}{a}(n+1)E_n(a) \end{aligned}$$

*with initial conditions*

$$(17) \quad A_0(a) = B_0(a) = E_0(a) = \frac{1}{a} \text{ and } C_0(a) = D_0(a) = 0.$$

These recursion can now be solved as in Section 3 to produce a closed-form expression for the integrals  $P_n(a)$  and  $Q_n(a)$ . We employ the notation

$$(18) \quad H_n = \sum_{k=1}^n \frac{1}{k}$$

for the harmonic numbers and

$$(19) \quad \text{Exp}_n(x) = \sum_{k=0}^n \frac{x^k}{k!}$$

for the partial sums of the exponential function.

**Theorem 4.5.** *Let  $a \in \mathbb{R}$  and  $n \in \mathbb{N}$ . Then*

$$(20) \quad P_n(a) := \int_0^1 x^n e^{-ax} dx = \frac{n!}{a^{n+1}} [1 - e^{-a} \text{Exp}_n(a)],$$

and

$$Q_n(a) := \int_0^1 x^n e^{-ax} \ln x dx = \frac{n!}{a^{n+1}} \left[ H_n - G(a) - e^{-a} \sum_{k=1}^n \frac{1}{k} \text{Exp}_{k-1}(a) \right],$$

where  $G(a) = -aQ_0(a) = \gamma + \Gamma(0, a) + \ln a$ .

These expressions provide the evaluations of two integrals in [4].

**Example 4.6.** Formula 4.351.2 states that

$$(21) \quad \int_0^1 e^{-ax}(-ax^2 + 2x) \ln x \, dx = \frac{1}{a^2} [-1 + (1+a)e^{-a}].$$

In order to verify this, observe that the stated integral is

$$(22) \quad -a \int_0^1 x^2 e^{-ax} \ln x \, dx + 2 \int_0^1 x e^{-ax} \ln x \, dx = -aQ_2(a) + 2Q_1(a).$$

The expressions in Theorem 4.5 now complete the evaluation.

**Example 4.7.** Formula 4.353.3 in [4] gives the value of

$$(23) \quad I_n(a) := \int_0^1 (-ax + n + 1)x^n e^{-ax} \ln x \, dx.$$

Observe that

$$(24) \quad I_n(a) = -aQ_{n+1}(a) + (n+1)Q_n(a),$$

and using the recursion (12) we conclude that  $I_n(a) = -P_n(a)$ . The expression in Theorem 4.5 is precisely what appears in [4].

We conclude with the evaluation of a series shown to us by Tewodros Amdeberhan. Expand the exponential term in (21) and integrate term by term to obtain

$$(25) \quad \sum_{k=0}^{\infty} \frac{(-a)^k}{k!(n+1+k)^2} = \frac{n!}{a^{n+1}} \left( -\psi(n+1) + \ln a + \Gamma(0, a) + e^{-a} \sum_{k=0}^n \frac{1}{k} \text{Exp}_{k-1}(a) \right).$$

Here

$$(26) \quad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$$

is the *digamma* function defined in 8.360.1 of [4]. the identity

$$(27) \quad \psi(n+1) = H_n - \gamma,$$

that is a direct consequence of the functional equation  $\Gamma(x+1) = x\Gamma(x)$  and  $\Gamma'(1) = -\gamma$ , was used to transform (25).

The identity (25) can be used to provide multiple expressions for the incomplete gamma function, such as

$$(28) \quad \int_a^{\infty} \frac{e^{-x}}{x} \, dx = \sum_{k=0}^{\infty} \frac{(-1)^k a^{n+1+k}}{n! k! (n+1+k)^2} + \psi(n+1) - \ln a - e^{-a} \sum_{k=1}^n \frac{\text{Exp}_{k-1}(a)}{k},$$

and the special case for  $n = 0$ :

$$(29) \quad \int_a^{\infty} \frac{e^{-x}}{x} \, dx = -\gamma - \ln a + \sum_{k=0}^{\infty} \frac{(-1)^k a^{k+1}}{(k+1)! (k+1)}.$$

These issues will be explored in a future publication.



## REFERENCES

- [1] T. Amdeberhan, L. Medina, and V. Moll. The integrals in Gradshteyn and Ryzhik. Part 5: Some trigonometric integrals. *Scientia*, to appear.
- [2] T. Amdeberhan and V. Moll. The integrals in Gradshteyn and Ryzhik. Part 7: Elementary examples. *Scientia*, to appear.
- [3] G. Boros and V. Moll. *Irresistible Integrals*. Cambridge University Press, New York, 1st edition, 2004.
- [4] I. S. Gradshteyn and I. M. Ryzhik. *Table of Integrals, Series, and Products*. Edited by A. Jeffrey and D. Zwillinger. Academic Press, New York, 7th edition, 2007.

DEPARTMENT OF MATHEMATICS, TULANE UNIVERSITY, NEW ORLEANS, LA 70118  
*E-mail address:* `vhm@math.tulane.edu`

DEPARTMENT OF MATHEMATICS, TULANE UNIVERSITY, NEW ORLEANS, LA 70118  
*E-mail address:* `jrosenbe@tulane.edu`

DEPARTMENT OF MATHEMATICS, TULANE UNIVERSITY, NEW ORLEANS, LA 70118  
*E-mail address:* `astraub@math.tulane.edu`

DEPARTMENT OF MATHEMATICS, TULANE UNIVERSITY, NEW ORLEANS, LA 70118  
*E-mail address:* `pwhitwor@tulane.edu`